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Some remarks on the extension of loop groups†

Bo-Yu Hou and Yao-Zhong Zhang
Institute of Modern Physics, Northwest University, Xian, China

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Abstract. It is shown that in an anomalous Yang-Mills theory in $1+1$ dimensions one can realise a non-central extension of a loop group, in addition to the well known central extension. The descent equations and Mickelsson's local coordinate techniques are used to construct 2-cocyles for the extended groups. It appears that in two-dimensional theories the central and non-central extensions of a loop group are compatible and equivalent to each other. The transition functions between different coordinate systems are also defined.

1. Introduction

Recently, there has been much active interest in certain infinite-dimensional Lie algebras in mathematics and theoretical physics (Fubini *et al* 1973, Friedan *et al* 1984, Goddard 1985, Olive 1985, Goddard and Horsley 1976, Kac 1980, Frenkel and Kac 1984). An example of an infinite-dimensional Lie algebra is the Kac-Moody algebra. The Kac-Moody algebra, which is also called a loop algebra, arises in non-Abelian anomalous gauge theories in two spacetime dimensions. The central extension of this algebra is called a commutator anomaly ('Schwinger-Jackiw-Johnson term') (Schwinger 1959, Jackiw and Johnson 1969) and is found to relate to the 2-cocyle of the cohomology of the Lie algebra of the group of gauge transformations (Faddeev 1984, Mickelsson 1983, 1985a, Nelson and Alvarez-Gaumé 1985). On the other hand, Pressley and Segal (1980) and Segal (1981) examine the central extension of the loop group corresponding to the loop algebra. They note that there is a topological obstruction to the construction of the extended group, i.e. one cannot think of the extended group as a product, but one should define it as a non-trivial fibre bundle. In a recent paper, Mickelsson (1985b) gives a concrete realisation of the central extension of the loop group in terms of local coordinate systems.

It is well known that in an anomalous Yang-Mills theory the anomalies entering the 'descent equations' (Stora 1983, Zumino 1983) are not uniquely defined. In particular, in one space dimension the commutator anomaly can be taken to be either 'c-number' (Zumino 1985) or to depend on a gauge field A (Zumino 1983); the former is the well known central extension of the loop algebra and the latter says that the Kac-Moody extension is non-central. On the group level, this means that we can construct two kinds of extensions of the loop group Γ: the central extension $\mathrm{U}(1)\to\hat{\Gamma}\to\Gamma$ and the non-central extension $\mathscr{A}(\mathrm{U}(1))\to\hat{\Gamma}\to\hat{\Gamma}$, where $\mathscr{A}(\mathrm{U}(1))$ is the group of functions $\lambda:\mathscr{A}\to\mathrm{U}(1)$ with pointwise multiplication and $\mathscr{A}$ is the space of all gauge fields.

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One object of this paper is to describe the central extension $U(1)\to\hat{\Gamma}\to\Gamma$ which exists in an anomalous Yang-Mills theory in two dimensions. At this point, we should emphasise that we are considering a different model from the Wess-Zumino model (Wess and Zumino 1971, Witten 1983) which Mickelsson considers and in which there is no coupling with the external gauge field. Then we shall realise explicitly the non-central extension $\mathscr{A}(U(1))\to\hat{\Gamma}\to\Gamma$ involved. We find that two kinds of extensions are compatible and equivalent to each other. It follows that in $(1+1)$-dimensional anomalous Yang-Mills theory we can realise both central and non-central extensions.

2. The central extension of the loop group

Let G be a compact Lie group and $\mathscr{G}$ its Lie algebra. Let $\langle a, b\rangle$ be an invariant bilinear form on $\mathscr{G}$. In one space dimension the simplest choice of commutator anomaly of an anomalous Yang-Mills theory gives rise to the current algebra which has the form of the Kac-Moody algebra $\hat{\mathscr{G}}=\mathrm{Map}(S^1,\mathscr{G})\oplus iR$ ($\mathrm{Map}(S^1, M)=$ the loop space of C^∞ maps: $S^1\to M$) with the commutator (Zumino 1985, Faddeev 1984)

$$[u, v](x)=[u(x), v(x)]+ic\int_{S^1}\langle u, u'\rangle\, dx \tag{2.1}$$

in which $u, v\in \mathrm{Map}(S^1,\mathscr{G})$, c is a real constant and iR commutes with everything. We want to construct a group $\hat{\Gamma}$ which has $\hat{\mathscr{G}}$ as its Lie algebra. We assume the following composition law (Segal 1981, Mickelsson 1985b)

$$(g_1,\lambda)(g_2,\mu)=(g_1g_2, \lambda\mu\exp[2\pi i\hat{\alpha}(g_1, g_2)]) \tag{2.2}$$

where $\lambda,\mu\in U(1)$ (circle group), $g_1, g_2\in\Gamma=\mathrm{Map}(S^1, G)$ (loop group) and $\hat{\alpha}$ is a real-valued function of g_1 and g_2; the product g_1g_2 is defined pointwise.

Let us now construct the $\hat{\alpha}$. For simplicity, we consider the case $G=SU(2)$. Locally the cocycle $\hat{\alpha}$ can be obtained from the so-called descent equations. The starting point is the Chern-Pontryagian density in four dimensions (S^4):

$$\Omega_4^{-1}(F)=-\frac{1}{8\pi^2}\operatorname{tr}F^2 \qquad \text{with } F=dA+A^2. \tag{2.3}$$

The four-dimensional integral of $\Omega_4^{-1}(F)$ is the second Chern number $\mathbb{Z}$. Let i specify a particular gauge, $i=A^{g_1\cdots g_i}=(g_1\ldots g_i)^{-1}A(g_1\ldots g_i)+(g_1\ldots g_i)^{-1}d(g_1\ldots g_i)$. Locally, Ω_4^{-1} is exact:

$$\Omega_4^{-1}(F)=d\Omega_3^0(A)\equiv d\Omega_3^0(0) \tag{2.4}$$

where

$$\Omega_3^0(0)=\frac{-1}{8\pi^2}\operatorname{tr}(AdA+\tfrac{2}{3}A^3) \tag{2.5}$$

is the Chern-Simons density. Taking the coboundary Δ (Faddeev 1984) of (2.4) gives $d\Delta\Omega_3^0(0)=0$ since Ω_4^{-1} is gauge invariant and d commutes with Δ. Thus locally

$$(\Delta\Omega_3^0)(0, 1)\equiv\Omega_3^0(1)-\Omega_3^0(0)=d\Omega_2^1(0, 1). \tag{2.6}$$

Using $\Delta^2=0$, we have from (2.6) $d\Delta\Omega_2^1=0$, i.e. $\Delta\Omega_2^1$ is exact:

$$(\Delta\Omega_2^1)(0,1,2)\equiv\Omega_2^1(1,2)-\Omega_2^1(0,2)+\Omega_2^1(0,1)=d\Omega_1^2(0,1,2). \tag{2.7}$$

Similarly $\Delta\Omega_1^2$ is exact

$$(\Delta\Omega_1^2)(0,1,2,3)\equiv\Omega_1^2(1,2,3)-\Omega_1^2(0,2,3)+\Omega_1^2(0,1,3)-\Omega_1^2(0,1,2)=d\Omega_0^3(0,1,2,3). \tag{2.8}$$

Equations (2.4)-(2.8) are called the descent equations (Stora 1983, Zumino 1983, Faddeev 1984).

Now let us consider (2.6) to be restricted to three dimensions (S^3). Restricting the Ω_2^1 so obtained to two dimensions (S^2), we get the non-Abelian anomaly Ω_2^1 in two dimensions. Similarly, consider (2.7) to be restricted to two dimensions and then restrict Ω_1^2 to one dimension (S^1); then we can define cochain

$$\hat{\alpha}=\int_{S^1}\Omega_1^2. \tag{2.9}$$

Now the cocyle $\hat{\alpha}$ can be defined as follows. The coboundary of Ω_3^0 is

$$(\Delta\Omega_3^0)(0,1)\equiv\Omega_3^0(1)-\Omega_3^0(0)=-\frac{1}{8\pi^2}d\,\mathrm{tr}(Adgg^{-1})+\frac{1}{24\pi^2}\mathrm{tr}(dgg^{-1})^3. \tag{2.10}$$

The last term $C^{(3)}=(1/24\pi^2)\,\mathrm{tr}(dgg^{-1})^3$ is closed and thus locally $C^{(3)}=dH^{(2)}$ for some 2-form $H^{(2)}$. In order to compute $H^{(2)}$, we consider the set $B=\{X\in\mathscr{G}|\frac{1}{2}\mathrm{tr}\,X^2<\pi^2\}$ and $\mathrm{G}_1=\mathrm{SU}(2)/\{-1\}$; then the exponential map $B\to\mathrm{G}_1$ is $1-1$ and C^∞; the sphere $\mathrm{tr}\,X^2=2\pi^2$ is mapped onto the point -1. Let g be a map from a 2-sphere in four dimensions into G_1; we can define the logarithm $\ln g:S^2\to B$. Write such g as $\exp u$ and we can compute $H^{(2)}$ from the integral

$$H^{(2)}(\ln g)=\frac{1}{24\pi^2}\int_0^1\mathrm{tr}(dg(x,t)g(x,t)^{-1})^3 \tag{2.11}$$

in which $g(x,t)=\exp(tu(x))$. By (2.6), (2.10) and (2.11), we obtain the non-Abelian anomaly in two dimensions:

$$\Omega_2^1(0,1)\equiv\Omega_2^1(\tilde{A},\tilde{A}^{\tilde{g}})=-\frac{1}{8\pi^2}\mathrm{tr}(\tilde{A}d\tilde{g}\tilde{g}^{-1})+H^{(2)}(\ln\tilde{g}) \tag{2.12}$$

where we have used $\tilde{Q}$ to denote a quantity Q to be restricted to two dimensions.

Applying the coboundary operation Δ to (2.12) gives

$$(\Delta\Omega_2^1)(0,1)\equiv\Omega_2^1(1,2)-\Omega_2^1(0,2)+\Omega_2^1(0,1)$$
$$=-\frac{1}{8\pi^2}\mathrm{tr}(\tilde{g}_1^{-1}d\tilde{g}_1d\tilde{g}_2\tilde{g}_2^{-1})+H^{(2)}(\ln\tilde{g}_1)+H^{(2)}(\ln\tilde{g}_2)-H^{(2)}(\ln\tilde{g}_1\tilde{g}_2). \tag{2.13}$$

We see that $\Delta\Omega_2^1$ does not depend on $\tilde{A}$; this is a property of the descent equations special to two-dimensional theories.

Let D^2 be some two-dimensional disc in four dimensions. Then any $\tilde{g}:D^2\to\mathrm{G}_1$ can be restricted to a map $g:\partial D^2\to\mathrm{G}_1$, where ∂D^2 is the boundary of D^2. We shall consider the case $S^1=\partial D^2$. Let $\tilde{g}_1$ and $\tilde{g}_2$ be two discs in G_1 such that $\tilde{g}_1\tilde{g}_2: D^2\to G_1$. We can define

$$\hat{\alpha}(g_1,g_2)=\int_{D^2}(\Delta\Omega_2^1)(0,1,2)=-\frac{1}{8\pi^2}\int_{D^2}\mathrm{tr}(\tilde{g}_1^{-1}d\tilde{g}_1 d\tilde{g}_2\tilde{g}_2^{-1})$$
$$+\int_{D^2}H^{(2)}(\ln\tilde{g}_1)+H^{(2)}(\ln\tilde{g}_2)-H^{(2)}(\ln\tilde{g}_1\tilde{g}_2). \tag{2.14}$$

The function $\hat{\alpha}(g_1,g_2)$ depends on the chosen restriction; in other words $\hat{\alpha}$ depends on the chosen disc D^2. The proof is as follows. Let $\tilde{g}_{1+}$, $\tilde{g}_{2+}: D^2_+\to G_1$ and $\tilde{g}_{1-}$, $\tilde{g}_{2-}: D^2_-\to G_1$ be the maps defined on the north semisphere D^2_+ and south semisphere D^2_-, respectively, such that $\tilde{g}_{1+}\tilde{g}_{2+}: D^2_+\to G_1$ and $\tilde{g}_{1-}\tilde{g}_{2-}: D^2_-\to G_1$; $(\tilde{g}_{1+},\tilde{g}_{2+})$ have the same values as $(\tilde{g}_{1-},\tilde{g}_{2-})$ when restricted on $S^1=\partial D^2_+=-\partial D^2_-$. Then

$$\hat{\alpha}(g_{1+},g_{2+})-\hat{\alpha}(g_{1-},g_{2-})=\int_{D^2_+}(\Delta\Omega_2^1)(0,1,2)-\int_{D^2_-}(\Delta\Omega_2^1)(0,1,2)$$
$$=-\frac{1}{8\pi^2}\int_{S^2}\mathrm{tr}(\tilde{g}_1^{-1}d\tilde{g}_1 d\tilde{g}_2\tilde{g}_2^{-1})+\int_{S^2}H^{(2)}(\ln\tilde{g}_1)+H^{(2)}(\ln\tilde{g}_2)-H^{(2)}(\ln\tilde{g}_1\tilde{g}_2)$$
$$\equiv\int_{S^2}(\Delta\Omega_2^1)(0,1,2). \tag{2.15}$$

We would like to show that (2.15) equals an integer $\mathbb{Z}$. For this purpose we divide S^4 into three discs $D^4(i)$, $i=0,1,2$ (Hou *et al* 1986). The boundary of $D^4(i)$ is a three-dimensional sphere $S^3(i)$, $\partial D^4(i)=S^3(i)$. We then divide $S^3(i)$ into sums of discs $D^3(i,j)$, $S^3(i)=\Sigma_{j(\neq i)}D^3(i,j)$, such that $D^3(i,j)$ have a common boundary S^2, $\partial D^3(i,j)=S^2(i,j)=(-1)^{i+j-1}S^2$. Then the last term in (2.15) is

$$\int_{S^2}(\Delta\Omega_2^1)(0,1,2)=\sum_{i<j}\int_{S^2(i,j)}\Omega_2^1(i,j)=\sum_{i<j}\int_{D^3(i,j)}\mathrm{d}\Omega_2^1(i,j)=\sum_{i<j}\int_{D^3(i,j)}(\Delta\Omega_3^0)(i,j)$$
$$=\sum_{i\neq j}\int_{D^3(i,j)}\Omega_3^0(i)=\sum_{i=0}^{2}\int_{S^3(i)}\Omega_3^0(i)=\sum_{i=0}^{2}\int_{D^4(i)}\mathrm{d}\Omega_3^0(i)$$
$$=\sum_{i=0}^{2}\int_{D^4(i)}\Omega_4^{-1}(F)=\int_{S^4}\Omega_4^{-1}(F)=\mathbb{Z}. \tag{2.16}$$

It follows from (2.15) and (2.16) that the function $\hat{\alpha}$ depends on the chosen restriction. However, the difference $\mathbb{Z}$ does not matter since we are only interested in $\exp(2\pi\mathrm{i}\hat{\alpha})$ and not in $\hat{\alpha}$ itself.

The new product law (2.2) is associative iff

$$\hat{\alpha}(g_2,g_3)+\hat{\alpha}(g_1,g_2g_3)-\hat{\alpha}(g_1,g_2)-\hat{\alpha}(g_1g_2,g_3)=\mathbb{Z}. \tag{2.17}$$

However, this is just the cocycle condition. In order to check (2.17), we notice that for the chosen restriction we can write by (2.7) and (2.14)

$$\hat{\alpha}(g_1,g_2)=\int_{S^1}\Omega_1^2(0,1,2). \tag{2.18}$$

Thus

$$(\Delta\hat{\alpha})(g_1,g_2,g_3)=\int_{S^1}(\Delta\Omega_1^2)(0,1,2,3). \tag{2.19}$$

Repeating the division procedure discussed above, $S^4 = \Sigma_{i=0}^3 D^4(i)$, $\partial D^4(i) = S^3(i) = \Sigma_{j(\neq i)} D^3(i,j)$, $\partial D^3(i,j) = S^2(i,j) = \Sigma_{k(\neq i,j)} D^2(i,j,k)$, such that $D^2(i,j,k)$ have a common boundary S^1, $\partial D^2(i,j,k) = S^1(i,j,k) = (-1)^{i+j+k-1} S^1$ $(i<j<k)$, we have

$$\begin{aligned}(\Delta\hat{\alpha})(g_1, g_2, g_3) &= \int_{S^1} (\Delta\Omega_1^2)(0,1,2,3) = \sum_{i<j<k} \int_{S^1(i,j,k)} \Omega_1^2(i,j,k) \\ &= \sum_{i<j<k} \int_{D^2(i,j,k)} \mathrm{d}\Omega_1^2(i,j,k) \\ &= \sum_{i<j<k} \int_{D^2(i,j,k)} (\Delta\Omega_2^1)(i,j,k) = \sum_{i<j\neq k} \int_{D^2(i,j,k)} \Omega_2^1(i,j) = \sum_{i<j} \int_{S^2(i,j)} \Omega_2^1(i,j) \\ &= \sum_{i<j} \int_{D^3(i,j)} \mathrm{d}\Omega_2^1(i,j) = \sum_{i<j} \int_{D^3(i,j)} (\Delta\Omega_3^0)(i,j) = \sum_{i\neq j} \int_{D^3(i,j)} \Omega_3^0(i) \\ &= \sum_{i=0}^3 \int_{S^3(i)} \Omega_3^0(i) = \sum_{i=0}^3 \int_{D^4(i)} \mathrm{d}\Omega_3^0(i) = \sum_{i=0}^3 \int_{D^4(i)} \Omega_4^{-1}(F) = \int_{S^4} \Omega_4^{-1}(F) = \mathbb{Z}.\end{aligned} \tag{2.20}$$

Thus equation (2.17) is seen to hold.

The function $\hat{\alpha}(g_1, g_2)$ defined above is only the 2-cocycle for the extended group of $\mathrm{Map}(S^1, \mathrm{G}_1)$ by U(1). The set $\mathrm{Map}(S^1, \mathrm{G}_1)$ is not a subgroup of Γ and one cannot continuously extend $\hat{\alpha}$ to the whole group because of the discontinuities arising from the logarithm. Therefore, in order to cover the whole group Γ one has to use local coordinate systems (Mickelsson 1985b). Choose two maps $\tilde{g}_1 : D^2 \to \mathrm{SU}(2)/\{-a\} = \mathrm{G}_a$, $\tilde{g}_2 : D^2 \to \mathrm{SU}(2)/\{-b\} = \mathrm{G}_b$ such that $\tilde{g}_1\tilde{g}_2 : D^2 \to \mathrm{SU}(2)/\{-c\} = \mathrm{G}_c$; then their restrictions on $S^1 = \partial D^2$ are maps $g_1 : S^1 \to \mathrm{G}_a$ and $g_2 : S^1 \to \mathrm{G}_b$. Obviously, for each $a \in \mathrm{SU}(2)$ the set of all maps $g : S^1 \to \mathrm{G}_a$ defines a coordinate patch V_a on $\mathrm{Map}(S^1, \mathrm{SU}(2))$; the sum of all coordinate patches V_a gives a cover of $\mathrm{Map}(S^1, \mathrm{SU}(2))$,

$$\mathrm{Map}(S^1, \mathrm{SU}(2)) = \bigcup_{a\in \mathrm{SU}(2)} V_a. \tag{2.21}$$

Thus we can define (Mickelsson 1985b)

$$\hat{\alpha}_{abc}(g_1, g_2) = -\frac{1}{8\pi^2} \int_{D^2} \mathrm{tr}(\tilde{g}_1^{-1} d\tilde{g}_1 d\tilde{g}_2 \tilde{g}_2^{-1}) + \int_{D^2} H^{(2)}(\ln a^{-1}\tilde{g}_1) + H^{(2)}(\ln b^{-1}\tilde{g}_2) - H^2(\ln C^{-1}\tilde{g}_1\tilde{g}_2). \tag{2.22}$$

Modulo integer $\mathbb{Z}$ the function $\hat{\alpha}_{abc}(g_1, g_2)$ does not depend on the chosen restriction. The proof is the same as the case in which $a = b = c = 1$ since $dH^{(2)}(\ln a^{-1}g) = C^{(3)}(a^{-1}g) = C^{(3)}(g)$ for a constant $a \in \mathrm{SU}(2)$ and the descent equations remain unchanged. The transition functions between V_a and V_b (for a gauge field A) are defined by

$$h_{ab}(g) = \int_{D^2} H^{(2)}(\ln a^{-1}\tilde{g}) - H^{(2)}(\ln b^{-1}\tilde{g}) \tag{2.23}$$

where $g \in V_a \cap V_b$ is a restriction of $\tilde{g} : D^2 \to \mathrm{G}_a \cap \mathrm{G}_b$ on $S^1 = \partial D^2$. The value $h_{ab}(g)$ does not depend on the chosen restriction since one is only interested in $\exp(2\pi i h_{ab})$. namely, let $g_\pm$ be two discs D_+^2 and D_-^2 in $\mathrm{G}_a \cap \mathrm{G}_b$ such that their restrictions on $S^1 = \partial D_+^2 = -\partial D_-^2$ have the same values. Then we can form $\tilde{g} : S^2 = D_+^2 \cup D_-^2 \to \mathrm{G}_a \cap \mathrm{G}_b$

by joining along the common boundary and the difference between the two expressions for h_{ab} is

$$
\begin{aligned}
h_{ab}(g_+) - h_{ab}(g_-) &= \int_{D_+^2} H^{(2)}(\ln a^{-1}\tilde{g}_+) - H^{(2)}(\ln b^{-1}\tilde{g}_+) \\
&\quad - \int_{D_-^2} H^{(2)}(\ln a^{-1}\tilde{g}_-) - H^{(2)}(\ln b^{-1}\tilde{g}_-) \\
&= \int_{S^2} H^{(2)}(\ln a^{-1}\tilde{g}) - H^{(2)}(\ln b^{-1}\tilde{g}) = \int_{D^3} C^{(3)}(g_a) - C^{(3)}(g_b) \\
&= \int_{S^3} C^{(3)}(g_{ab}) = W[g_{ab}] = \mathbb{Z}
\end{aligned}
\tag{2.24}
$$

where $g_{ab} : S^3 \to \mathrm{G}$ is obtained from g_a and g_b by joining along $S^2 = \partial D^3$ and $W[g_{ab}]$ stands for the winding number of map g_{ab}. Therefore by the definitions

$$
\hat{\alpha}_{abc}(g_1, g_2) - \hat{\alpha}_{a'bc}(g_1, g_2) = h_{aa}(g_1)
\tag{2.25}
$$

and similarly for the indices b and c.

3. The non-central extension of the loop group

As is well known, in one space dimension the current algebra of an anomalous Yang–Mills theory can have the following anomaly (Zumino 1983):

$$
[u, v](x) = [u(x), v(x)] + \mathrm{i}C \int_{S^1} \mathrm{tr}(uvA)
\tag{3.1}
$$

where A is a $\mathscr{G}$-valued vector potential 1-form on S^1. The Lie algebra extension of $\mathrm{Map}(S^1, \mathscr{G})$ is now infinite dimensional and thus the extension is non-central. It is natural to think of (3.1) as the Kac–Moody algebra on the gauge orbit through A in the space of all vector potentials; the different gauge orbits correspond to the different Kac-Moody algebras, in contrast to the case discussed in § 2. In order to examine the group corresponding to (3.1), we consider the extension $\mathscr{A}(\mathrm{U}(1)) \to \hat{\Gamma} \to \Gamma$; it leads us to assume the multiplication table

$$
(g_1, \lambda)(g_2, \mu) = (g_1 g_2, \lambda\mu \exp[2\pi \mathrm{i}\alpha(A; g_1, g_2)])
\tag{3.2}
$$

where $\lambda, \mu \in \mathscr{A}(\mathrm{U}(1))$, and now α is a real-valued function of g_1, g_2 and A. We can ask what the group cocycle α is. Following the same arguments as in the previous section, locally we can define

$$
\begin{aligned}
\alpha(A; g_1, g_2) &= \int_{D^2} (\Delta\Omega_2^1)(\tilde{A}, \tilde{A}^{\tilde{g}_1}, \tilde{A}^{\tilde{g}_1\tilde{g}_2}) - \int_{S^1} (\Delta X_1^1)(A, A^{g_1}, A^{g_1 g_2}) \\
&\equiv \hat{\alpha}(g_1, g_2) - \int_{S^1} X_1^1(A^{g_1}, A^{g_1 g_2}) - X_1^1(A, A^{g_1 g_2}) + X_1^1(A, A^{g_1})
\end{aligned}
\tag{3.3}
$$

where

$$
X_1^1(A, A^g) = \int_{\Delta^1} \chi_1^1
\tag{3.4}
$$

with Δ^1 being a 1-simplex in group space, and χ_1^1, given by the so-called 'triangle formula' (Manes *et al* 1986), has the following expression:

$$\chi_1^1 = \frac{1}{24\pi^2} \operatorname{tr}\{g^{-1}\delta g(g^{-1}Ag + g^{-1}dg)\} \tag{3.5}$$

in which δ is the differentiation with respect to any parameters which the element g may depend on. When g_1 and g_2 are in the vicinity of the unit element the function α reduces to the anomaly in equation (3.1).

Modulo integer $\mathbb{Z}$ the function α does not depend on the chosen extension; the proof is essentially the same as in the previous section. Notice that α differs by a coboundary of $\int_{S^1} X_1^1(A, A^g)$ from $\hat{\alpha}$; therefore α and $\hat{\alpha}$ are cohomologous, i.e. they define the equivalent extension of the loop group Γ (Segal 1981). It follows that the central extension and the non-central extension involved in $(1+1)$-dimensional anomalous Yang–Mills theories are actually equivalent to each other. This is true only in $(1+1)$-dimensional theories.

It is not hard to see that the function α satisfies the cocycle condition because of the nilpotency of the operation Δ,

$$\alpha(A^{g_1}; g_2, g_3) + \alpha(A; g_1, g_2g_3) - \alpha(A; g_1, g_2) - \alpha(A; g_1g_2, g_3) = \mathbb{Z} \tag{3.6}$$

which is also an expression for the associativity of the new multiplication law (3.2). In order to avoid the discontinuities arising from the logarithm and to cover the whole group Γ, one has to use again the local coordinate systems V_a defined in § 2. Choose two discs $\tilde{g}_1 : D^2 \to G_a$, $\tilde{g}_2 : D^2 \to G_b$ such that $\tilde{g}_1\tilde{g}_2 : D^2 \to G_c$; then we get their restrictions on $S^1 = \partial D^2$, $g_1 \in V_a$ and $g_2 \in V_b$. By (3.3) we can define

$$\alpha_{abc}(A; g_1, g_2) = \hat{\alpha}_{abc}(g_1, g_2) - \int_{S^1} X_1^1(A^{g_1}, A^{g_1g_2}) - X_1^1(A, A^{g_1g_2}) + X_1^1(A, A^{g_1}) \tag{3.7}$$

where $\hat{\alpha}_{abc}$ is given by (2.22). The transition functions h_{ab} between V_a and V_b (for a gauge field A) are still given by (2.23).

References

Faddeev L D 1984 *Phys. Lett.* **145B** 81
Frenkel I and Kac V G 1984 *Invent. Math.* **62** 455
Friedan D, Qui Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575
Fubini S, Hanson A J and Jackiw R 1973 *Phys. Rev.* D **7** 1732
Goddard P 1985 *DAMTP preprint* 85-7
Goddard P and Horsley R 1976 *Nucl. Phys.* B **111** 272
Hou B Y, Hou B Y and Wang P 1986 *Lett. Math. Phys.* **11** 179
Jackiw R and Johnson K 1969 *Phys. Rev.* **182** 1459
Kac 1980 *Adv. Math.* **35** 264
Manes J, Stora R and Zumino B 1986 *Commun. Math. Phys.* **102** 157
Mickelsson J 1983 *Lett. Math. Phys.* **7** 45
—— 1985a *Commun. Math. Phys.* **97** 361
—— 1985b *Phys. Rev. Lett.* **55** 301
Nelson P and Alvarez-Gaumé L 1985 *Commun. Math. Phys.* **99** 103
Olive D I 1985 *Preprint Imperial/TP/84-85/14*
Pressley A N and Segal G B 1980 *Loop groups* (Oxford: Oxford University Press)
Schwinger 1959 *Phys. Rev. Lett.* **3** 296

Segal G B 1981 *Commun. Math. Phys.* **80** 301
Stora R 1983 *Cargese Lectures* (New York: Plenum)
Wess J and Zumino B 1971 *Phys. Lett.* **37B** 95
Witten E 1983 *Nucl. Phys.* B **223** 455
Zumino B 1983 *Relativity, Groups and Topology* ed B S DeWitt and R Store (Amsterdam: North-Holland)
—— 1985 *Nucl. Phys.* B **253** 477